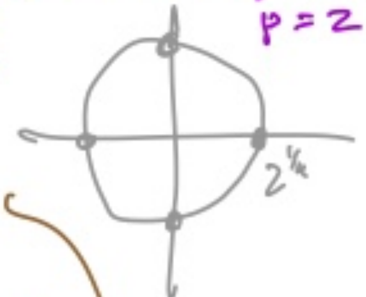
$\boxed{\text{Example}}$ Find $G(E/\mathbb{Q})$, where $E = \mathbb{Q}(\sqrt[4]{2}, i)$. Show that FTGT applies, and find the correspondences between subfields of E & subgroups of $G(E/\mathbb{Q})$.

Solution:

- Minimal polynomial of i is x^2+1. (irred over $\mathbb{Q}$) rational roots test.
  ↪ roots $i, -i$.
- Minimal polynomial of $\sqrt[4]{2}$ is x^4-2 (irred over $\mathbb{Q}$ by Eisenstein $p=2$)

roots of x^4-2 are $\pm 2^{1/4}, \pm i 2^{1/4}$



$\Rightarrow$ (splitting field of x^4-2) $\subseteq E$.

Also $E \subseteq$ (splitting field of x^4-2), because

$\sqrt[4]{2} \in$ splitting field of x^4-2,

$i = i2^{1/4} / 2^{1/4} \in$ splitting field,

$\Rightarrow E =$ splitting field.

$$E = \{c_0 + c_1\sqrt[4]{2} + c_2\sqrt{2} + c_3 2^{3/4} + c_4 i + c_5 i\sqrt[4]{2} + c_6 i\sqrt{2} + c_7 i 2^{3/4}\} : c_j \in \mathbb{Q}\ \forall j$$

$E = (\mathbb{Q}[\sqrt[4]{2}])(i)$

$[E:\mathbb{Q}] = [E:\mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}] = 2\cdot 4 = 8.$ ✓

Let's now compute the automorphisms in $G(E/\mathbb{Q})$.

They are determined by what happens to $\sqrt[4]{2}$, what happens to i.

Options: $\sqrt[4]{2} \to \{\underset{1}{\sqrt[4]{2}}, \underset{2}{-\sqrt[4]{2}}, \underset{3}{i\sqrt[4]{2}}, \underset{4}{-i\sqrt[4]{2}}\}$

$i \to i, -i.$

Let σ be defined by $\sigma(\sqrt[4]{2}) = -\sqrt[4]{2}$, $\sigma(i) = i$

$\sqrt[4]{2} \to -\sqrt[4]{2}$, $-\sqrt[4]{2} \to \sqrt[4]{2}$, $i\sqrt[4]{2} \leftrightarrow -i\sqrt[4]{2}$ $(1,2)(3,4)$

Let τ be defined by $\tau(\sqrt[4]{2}) = \sqrt[4]{2}$, $\tau(i) = -i$

$\sqrt[4]{2} \to \sqrt[4]{2}$, $-\sqrt[4]{2} \to -\sqrt[4]{2}$, $i\sqrt[4]{2} \leftrightarrow -i\sqrt[4]{2}$ $(3,4)$

$\sigma^2 = e$
$\tau^2 = e$ $\Rightarrow$ $\sigma\tau = \tau\sigma$

$(1,2)$ $\sqrt[4]{2} \leftrightarrow -\sqrt[4]{2}$, $\pm i\sqrt[4]{2}$

Let η be defined by

$$\eta(\sqrt[4]{2}) = i\sqrt[4]{2} \qquad (1\,3\,2\,4)$$
$$\eta(-\sqrt[4]{2}) = -i\sqrt[4]{2}$$
$$\eta(i\sqrt[4]{2}) = -\sqrt[4]{2}$$
$$\eta(-i\sqrt[4]{2}) = \sqrt[4]{2}$$

$$G(E/F) \cong \{e, \overset{\sigma\tau = \tau\eta^2}{(1,2)}, \overset{\tau}{(3,4)}, \overset{\sigma = \eta^2}{(1,2)(3,4)}, \overset{\eta}{(1\,3\,2\,4)}, \overset{\eta^3}{(1\,4\,2\,3)}, \overset{\sigma\tau\eta = \tau\eta^3}{(1\,3)(2\,4)}, \overset{\tau\eta}{(1\,4)(2\,3)}\}$$

$(1\,2)(1\,3\,2\,4) = (1\,3)(2\,4) = \tau\eta^3$

$(3\,4)(1\,3\,2\,4) = (1\,4)(2\,3) = \tau\eta$

$\eta^4 = e$, $\sigma^2 = \tau^2 = e$, $\sigma\tau = \tau\sigma$, $\sigma\eta = \eta\sigma$

$\sigma\eta \leftrightarrow (1\,2)(3\,4)(1\,3\,2\,4) = (1\,4\,2\,3) = \eta^3$

$\eta\sigma \leftrightarrow (1\,3\,2\,4)(1\,2)(3\,4) = (1\,4\,2\,3) = \eta^3$ ie $\sigma = \eta^2$ so $\eta^2\tau = \tau\eta^2$

$\tau\eta = (3\,4)(1\,3\,2\,4) = (1\,4)(2\,3) = \eta^3$

$\eta\tau = (1\,3\,2\,4)(3\,4) = (1\,3)(2\,4) = \sigma\tau\eta = \eta^2\tau\eta \overset{\tau\eta^3}{=} \Rightarrow \eta^3\tau = \tau\eta,$

$\Rightarrow \tau\eta^3 = \eta^2\tau\eta = \eta^2\eta^3\tau = \eta\tau$

$$\therefore\ G(E/F) \cong \{e, \eta, \eta^2, \eta^3, \tau, \underset{\eta^3\tau}{\overset{\shortparallel}{\tau\eta}}, \underset{\eta^2\tau}{\overset{\shortparallel}{\tau\eta^2}}, \underset{\eta\tau}{\overset{\shortparallel}{\tau\eta^3}}\}$$

$\tau^2 = e,\ \eta^4 = e$

$$\cong \langle \eta, \tau : \tau^2 = e, \eta^4 = e, \eta\tau = \tau\eta^3 \rangle$$

$$\cong D_4 \quad \text{(symmetries of square)}$$

Subgroups of D_4: $\{e\}$, $\langle\tau\rangle \cong \mathbb{Z}_2$, $\langle\eta^2\rangle \cong \mathbb{Z}_2$, $\langle\tau\eta^2\rangle \cong \mathbb{Z}_2$, $\langle\tau\eta^3\rangle \cong \mathbb{Z}_2$, $\langle\tau\eta\rangle \cong \mathbb{Z}_2$, $\langle\eta\rangle \cong \mathbb{Z}_4$, $\langle\tau, \eta^2\rangle \cong \mathbb{Z}_2\times\mathbb{Z}_2$, $\langle\tau\eta, \tau\eta^3\rangle \cong \mathbb{Z}_2\times\mathbb{Z}_2$, D_4

$\underset{a}{\tau\eta}, \underset{b}{\tau\eta^3}$:

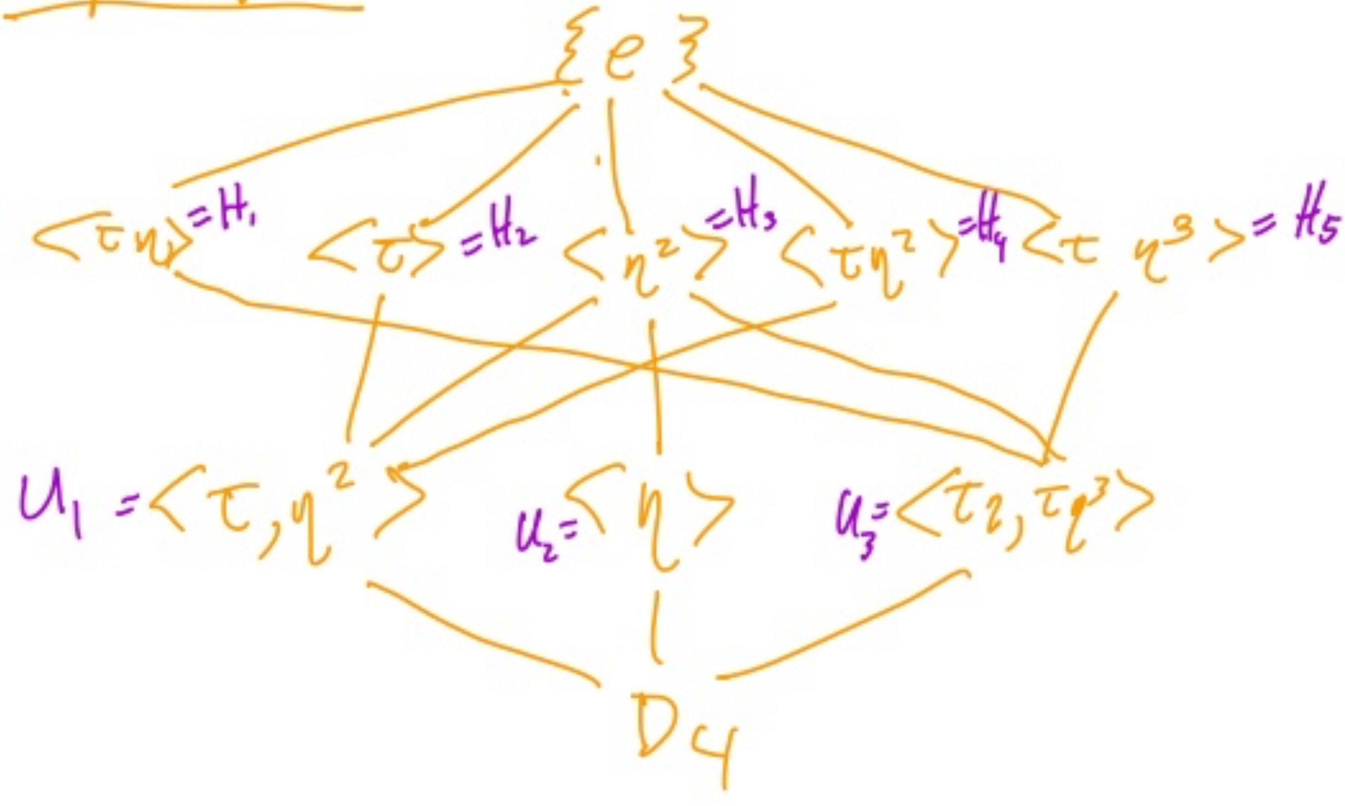
$a^2 = \tau\eta\tau\eta = \tau\eta\eta^3\tau = e$

$b^2 = \tau\eta^3\tau\eta^3 = \tau\eta^3\eta\tau = e$

$ab = \tau\eta\tau\eta^3 = \tau^2\eta^6 = \eta^2$

$ba = \tau\eta^3\tau\eta = \tau\eta^6\tau = \tau\eta^2\tau = \eta^2$

Group Diagram:



Fixed Fields

Recall $\tau: \begin{pmatrix} \sqrt[4]{2} \\ -\sqrt[4]{2} \\ i\sqrt[4]{2} \\ -i\sqrt[4]{2} \end{pmatrix} \to \begin{pmatrix} \sqrt[4]{2} \\ -\sqrt[4]{2} \\ -i\sqrt[4]{2} \\ i\sqrt[4]{2} \end{pmatrix}$

$$E = \{c_0 + c_1\sqrt[4]{2} + c_2\sqrt{2} + c_3 2^{3/4} + c_4 i + c_5 i\sqrt[4]{2} + c_6 i\sqrt{2} + c_7 i 2^{3/4} : c_j \in \mathbb{Q}\ \forall j\}.$$

$$\tau\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ -c_4 \\ -c_5 \\ -c_6 \\ -c_7 \end{pmatrix};$$

$\eta: \begin{pmatrix} \sqrt[4]{2} \\ -\sqrt[4]{2} \\ i\sqrt[4]{2} \\ -i\sqrt[4]{2} \end{pmatrix} \to \begin{pmatrix} i\sqrt[4]{2} \\ -i\sqrt[4]{2} \\ -\sqrt[4]{2} \\ \sqrt[4]{2} \end{pmatrix}$

$$\eta\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \begin{pmatrix} c_0 \\ -c_5 \\ -c_2 \\ c_7 \\ c_4 \\ c_1 \\ -c_6 \\ -c_3 \end{pmatrix}, \quad \eta^2: \begin{pmatrix} c_0 \\ \vdots \\ c_7 \end{pmatrix} \mapsto \begin{pmatrix} c_0 \\ -c_1 \\ c_2 \\ -c_3 \\ c_4 \\ -c_5 \\ c_6 \\ -c_7 \end{pmatrix}$$

$$\eta\left((-i\sqrt[4]{2})^3\right) = 2^{3/4}$$
$$\eta(-i^3 2^{3/4}) = 2^{3/4}$$
$$\eta(i2^{3/4}) = 2^{3/4}$$
$$\eta(2^{1/4}(-i)2^{1/4}) = (i2^{1/4}\cdot 2^{1/4}) = i\sqrt{2}$$
$$\overset{\|}{\eta}(-i\sqrt{2})$$
$$\eta(2^{1/4}(-i2^{1/2})) = -i2^{1/4}(2^{1/4})^2 = i2^{3/4}$$
$$\Rightarrow \eta(-2^{3/4}) = i2^{3/4}$$

$$\Rightarrow \tau\eta: \begin{pmatrix} c_0 \\ \vdots \\ c_7 \end{pmatrix} \mapsto \begin{pmatrix} c_0 \\ c_5 \\ -c_2 \\ -c_7 \\ -c_4 \\ c_1 \\ c_6 \\ -c_3 \end{pmatrix}$$

$$\Rightarrow \tau\eta^2: \begin{pmatrix} c_0 \\ \vdots \\ c_7 \end{pmatrix} \mapsto \begin{pmatrix} c_0 \\ -c_1 \\ c_2 \\ -c_3 \\ -c_4 \\ c_5 \\ -c_6 \\ c_7 \end{pmatrix}$$

$$\Rightarrow \eta\tau = \tau\eta^3: \begin{pmatrix} c_0 \\ \vdots \\ c_7 \end{pmatrix} \longmapsto \begin{pmatrix} c_0 \\ -c_5 \\ -c_2 \\ c_7 \\ -c_4 \\ -c_1 \\ c_6 \\ c_3 \end{pmatrix}$$

From this we get

$$E_{\{\tau\}} = E_{H_2} = \text{span}\{1, 2^{1/4}, 2^{1/2}, 2^{3/4}\} = \mathbb{Q}(2^{1/4})$$

$$E_{\{\eta\}} = E_{U_2} = \text{span}\{1, i\} = \mathbb{Q}(i)$$

$$E_{H_1} = E_{\{\tau\eta\}} = \text{span}\{1, \overset{c_1+c_5}{2^{1/4}+i2^{1/4}}, \overset{c_3-c_7}{2^{3/4}-i2^{3/4}}, \overset{c_6}{i\sqrt{2}}\}$$
$$= \mathbb{Q}((1+i)2^{1/4})$$

$$E_{H_3} = E_{\{\eta^2\}} = \text{span}\{1, \sqrt{2}, i, i\sqrt{2}\} = \mathbb{Q}(i, \sqrt{2})$$

$$E_{H_4} = E_{\{\tau\eta^2\}} = \text{span}\{1, \sqrt{2}, i2^{1/4}, i2^{3/4}\} = \mathbb{Q}(i2^{1/4})$$

$$E_{H_5} = E_{\{\tau\eta^3\}} = \text{span}\{1, \overset{c_3+c_7}{2^{3/4}+i2^{3/4}}, \overset{c_1-c_5}{2^{1/4}-i2^{1/4}}, \overset{c_6}{i\sqrt{2}}\} = \mathbb{Q}((1-i)2^{1/4})$$

$$E_{U_1} = E_{\{\tau, \eta^2\}} = \text{span}\{1, \sqrt{2}\} = \mathbb{Q}(\sqrt{2})$$

$$E_{U_3} = E_{\{\tau\eta, \tau\eta^3\}} = \text{span}\{1, i\sqrt{2}\} = \mathbb{Q}(i\sqrt{2}) \, .$$

Therefore, the field diagram is (corresponding to the groups)